

DICHOTOMY OF STABLE RADIAL SOLUTIONS OF $-\Delta u = f(u)$ OUTSIDE A BALL

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ABSTRACT. This paper is devoted to the study of stable radial solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N \setminus B_1 = \{x \in \mathbb{R}^N : |x| \geq 1\}$, where $f \in C^1(\mathbb{R})$ and $N \geq 2$. We prove that such solutions are either large [in the sense that $|u(r)| \geq Mr^{-N/2+\sqrt{N-1}+2}$, if $2 \leq N \leq 9$; $|u(r)| \geq M \log(r)$, if $N = 10$; $|u(r) - u_\infty| \geq Mr^{N/2+\sqrt{N-1}+2}$, if $N \geq 11$; $\forall r \geq r_0$, for some $M > 0$, $r_0 \geq 1$] or small [in the sense that $|u(r)| \leq M \log(r)$, if $N = 2$; $|u(r) - u_\infty| \leq Mr^{N/2-\sqrt{N-1}+2}$; if $N \geq 3$; $\forall r \geq 2$, for some $M > 0$], where $u_\infty = \lim_{r \rightarrow \infty} u(r) \in [-\infty, +\infty]$. These results can be applied to stable outside a compact set radial solutions of equations of the type $-\Delta u = g(u)$ in \mathbb{R}^N . We prove also the optimality of these results, by considering solutions of the form $u(r) = r^\alpha$ or $u(r) = \log(r)$, $\forall r \geq 1$, where $\alpha \in \mathbb{R} \setminus \{0\}$.

1. INTRODUCTION AND MAIN RESULTS

This paper deals with the stability of radial solutions of

$$(1.1) \quad -\Delta u = f(u) \quad \text{in } \mathbb{R}^N \setminus B_1,$$

where B_1 is the open unit ball of \mathbb{R}^N , $N \geq 2$ and $f \in C^1(\mathbb{R})$. We consider classical solutions $u \in C^2(\mathbb{R}^N \setminus B_1)$.

A solution u of (1.1) is called stable if

$$Q_u(v) := \int_{\mathbb{R}^N \setminus \overline{B_1}} (|\nabla v|^2 - f'(u)v^2) \, dx \geq 0$$

for every $v \in C^1(\mathbb{R}^N \setminus \overline{B_1})$ with compact support in $\mathbb{R}^N \setminus \overline{B_1}$. Note that the above expression is nothing but the second variation of the energy functional associated to (1.1) in a bounded domain $\Omega \subset \mathbb{R}^N \setminus \overline{B_1}$: $E_\Omega(u) = \int_\Omega (|\nabla u|^2/2 - F(u)) \, dx$, where $F' = f$. Thus, if $u \in C^1(\mathbb{R}^N \setminus B_1)$ is a local minimizer of E_Ω for every bounded smooth domain $\Omega \subset \mathbb{R}^N \setminus \overline{B_1}$ (i.e., a minimizer under every small enough $C^1(\overline{\Omega})$ perturbation vanishing on $\partial\Omega$), then u is a stable solution of (1.1).

We will be also interested in stable outside a compact set radial solutions of

$$(1.2) \quad -\Delta u = g(u) \quad \text{in } \mathbb{R}^N,$$

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where $N \geq 2$ and $g \in C^1(\mathbb{R})$.

We say that a classical solution $u \in C^2(\mathbb{R}^N)$ of (1.2) is stable outside a compact set if there exists a compact set $K \subset \mathbb{R}^N$ such that $Q_u(v) = \int_{\mathbb{R}^N} (|\nabla v|^2 - g'(u)v^2) dx \geq 0$ for every $v \in C^1(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N \setminus K$.

Clearly the stability outside a compact set of a solution of (1.2) is equivalent to the existence of $R_0 > 0$ such that u is stable in $\mathbb{R}^N \setminus B_{R_0}$. It follows easily that the function $w(x) := u(R_0 x)$ is an stable solution of $-\Delta w = R_0^2 g(w)$ in $\mathbb{R}^N \setminus B_1$ and we can apply the results obtained for such solutions.

On the other hand we say that a classical solution $u \in C^2(\mathbb{R}^N)$ of (1.2) has finite Morse index equal to an integer $k \geq 0$ if k is the maximal dimension of a subspace $X_k \subset C_c^1(\mathbb{R}^N)$ (the space of $C^1(\mathbb{R}^N)$ functions with compact support) such that

$$Q_u(\varphi) = \int_{\mathbb{R}^N} (|\nabla \varphi|^2 - g'(u)\varphi^2) dx < 0 \quad \text{for all } \varphi \in X_k \setminus \{0\}.$$

If there is no such finite integer k , we say that u has infinite Morse index.

Clearly, every stable solution has finite Morse index equal to 0. It is also easily seen that every solution with finite Morse index is stable outside a compact set. Indeed, if $X_k = \text{Span}\{\varphi_1, \dots, \varphi_k\}$ is a subspace of dimension k of $C_c^1(\mathbb{R}^N)$ such that $Q_u(\varphi) < 0$ for any $\varphi \in X_k \setminus \{0\}$ and $K := \bigcup_{j=1}^k \text{supp}(\varphi_j)$, then $Q_u(v) \geq 0$ for every $v \in C_c^1(\mathbb{R}^N \setminus K)$, and the claim is proved. Hence, we can apply to finite Morse index solutions the result obtained for solutions which are stable outside a compact set.

Farina [6, 7] studied the stability and stability outside a compact set of nontrivial solutions of the Lane-Emden equation $-\Delta u = |u|^{p-1}u$ in \mathbb{R}^N ($p > 1$). It is proved that the existence of such solutions depend on N (dimensions $N = 2$ and $N = 10$ are critical in some sense, as in the main results of this paper) and p (there are two critical values: $p = (N+2)/(N-2)$, the usual critical exponent in Sobolev imbedding theorems, defined for $N \geq 3$; and $p = p_c := ((N-2)^2 - 4N + 8\sqrt{N-1}) / ((N-2)(N-10))$, defined for $N \geq 11$). A complete classification of radial solutions of this equation which are stable outside a compact set is also given (see Remark 4 below).

Farina [8] considered the equation $-\Delta u = e^u$ in \mathbb{R}^N , obtaining that there are no stable solutions if $N \leq 9$. A complete classification of solutions of this equation which are stable outside a compact set is obtained for $N = 2$ (these solutions are radially symmetric, up to a translation). In a later paper, Dancer and Farina [3] studied also this equation and proved that there are no solutions which are stable outside a compact set if $3 \leq N \leq 9$ (again dimensions $N = 2$ and $N = 10$ are critical in some sense).

Dupaigne and Farina [4, 5] have also studied the stability and stability outside a compact set of equation (1.2), for a large class of functions $g \in$

$C^1(\mathbb{R})$. Among other things, they proved that if $g \geq 0$ and $1 \leq N \leq 4$, then there are no nonconstant bounded stable solutions.

We are interested in radial solutions of (1.1) and (1.2). By abuse of notation, we write $u(r)$ instead of $u(x)$, where $r = |x|$ and $x \in \mathbb{R}^N$. We will denote by u_r the radial derivative of a radial function u .

Concerning with nonconstant bounded stable radial solutions $u \in C^2(\mathbb{R}^N)$ of (1.2), Cabré and Capella [1] proved that there are no such solutions if $N \leq 10$ and g satisfies a nondegeneracy condition. The author [10] refined this result, proving that this nondegeneracy condition is not necessary and giving sharp pointwise estimates related to the asymptotic behavior of such solutions (not necessarily bounded). Specifically, in [10] it is proved that every nonconstant radial stable solution of (1.2) satisfies $|u(r)| \geq Mr^{-N/2+\sqrt{N-1}+2}$, if $N \neq 10$, and $|u(r)| \geq M \log(r)$, if $N = 10$; $\forall r \geq r_0$, for some $M, r_0 > 0$.

In this paper we establish that there is a dichotomy of radial stable solutions of (1.1): such solutions are either *large* (i.e. roughly speaking, they grow at least like a power $r^{-N/2+\sqrt{N-1}+2}$, like the mentioned stable solutions in all of \mathbb{R}^N) or *small* (i.e. roughly speaking, they grow at most like a power $r^{-N/2-\sqrt{N-1}+2}$). Note that the exponents $-N/2 \pm \sqrt{N-1} + 2$ vanish at $N = 2$ and at $N = 10$, respectively. Hence, we can consider that these dimensions are critical in the study of these problems.

Theorem 1.1. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$ and u be a radial stable solution of (1.1) (not necessarily bounded). Then there exists $u_\infty := \lim_{r \rightarrow \infty} u(r) \in [-\infty, +\infty]$ and u satisfies either (L) or (S). Here*

(L) *There exist $M > 0$, $r_0 \geq 1$ such that*

$$\begin{aligned} |u(r)| &\geq Mr^{-N/2+\sqrt{N-1}+2} & \forall r \geq r_0 \text{ if } 2 \leq N \leq 9, \\ |u(r)| &\geq M \log(r) & \forall r \geq r_0 \text{ if } N = 10, \\ |u(r) - u_\infty| &\geq Mr^{-N/2+\sqrt{N-1}+2} & \forall r \geq r_0 \text{ if } N \geq 11. \end{aligned}$$

(S) *There exists $M > 0$ such that*

$$\begin{aligned} |u(r)| &\leq M \log(r) & \forall r \geq 2 \text{ if } N = 2, \\ |u(r) - u_\infty| &\leq Mr^{-N/2-\sqrt{N-1}+2} & \forall r \geq 1 \text{ if } N \geq 3. \end{aligned}$$

Note that in the case $N \geq 11$, condition (L) is relevant if u is bounded. Otherwise $u_\infty = \pm\infty$ and the inequality in (L) is vacuous. On the other hand, if $N \geq 3$, condition (S) says implicitly that $u_\infty \in \mathbb{R}$ and hence u is bounded.

Example 3.1 below shows that the exponents $-N/2 \pm \sqrt{N-1} + 2$ which appear in Theorem 1.1 are optimal. In fact any pure power in the set $(-\infty, -N/2 - \sqrt{N-1} + 2] \cup [-N/2 + \sqrt{N-1} + 2, +\infty)$ is allowed for stable solutions (considering logarithm as a 0-power).

If $u \in C^2(\mathbb{R}^N)$ is a nonconstant radial stable solution of (1.2) then, by a result of the author ([10, Th. 2.1 and 2.2]), u satisfies (L). This is consistent with Theorem 1.1.

Theorem 1.2. *Let $N \geq 3$, $f \in C^1(\mathbb{R})$ and u be a bounded radial stable solution of (1.1). Then there exists $u_\infty := \lim_{r \rightarrow \infty} u(r) \in \mathbb{R}$ and u satisfies either (L') or (S'). Here*

(L') $N \geq 11$ and there exist $M > 0$, $r_0 \geq 1$ such that

$$|u(r) - u_\infty| \geq Mr^{-N/2 + \sqrt{N-1} + 2} \quad \forall r \geq r_0.$$

Moreover, $|\nabla u| \notin L^p(\mathbb{R}^N \setminus \overline{B_1})$ for every $1 \leq p \leq \frac{N}{N/2 - \sqrt{N-1} - 1}$.

In particular $|\nabla u| \notin L^2(\mathbb{R}^N \setminus \overline{B_1})$.

(S') There exists $M > 0$ such that

$$|u(r) - u_\infty| \leq Mr^{-N/2 - \sqrt{N-1} + 2} \quad \forall r \geq 1.$$

Moreover, $|\nabla u| \in L^p(\mathbb{R}^N \setminus \overline{B_1})$ for every $\frac{N}{N/2 + \sqrt{N-1} - 1} < p \leq \infty$.

In particular $|\nabla u| \in L^2(\mathbb{R}^N \setminus \overline{B_1})$.

Corollary 1.3. *Let $N \geq 2$, $g \in C^1(\mathbb{R})$ and u be a radial solution of (1.2) which is stable outside a compact set (not necessarily bounded). Then u satisfies either (L) or (S).*

Corollary 1.4. *Let $N \geq 3$, $g \in C^1(\mathbb{R})$ and u be a bounded radial solution of (1.2) which is stable outside a compact set. Then u satisfies either (L') or (S').*

To distinguish if a nonconstant radial stable solution of (1.1) is *large* or *small* we will consider the following properties:

(H_L) There exists $R_1 \geq 1$ such that

$$\int_{R_1}^{R_2} r^{N-1} u_r^2 \left(\eta'^2 - \frac{N-1}{r^2} \eta^2 \right) dr \geq 0,$$

for every $R_2 > R_1$ and $\eta \in C^{0,1}([R_1, R_2])$ such that $\eta(R_2) = 0$.

(H_S) For every $R_1 \geq 1$ there exist $R_2 > R_1$ and $\eta_0 \in C^{0,1}([R_1, R_2])$ such that $\eta_0(R_2) = 0$ and

$$\int_{R_1}^{R_2} r^{N-1} u_r^2 \left(\eta_0'^2 - \frac{N-1}{r^2} \eta_0^2 \right) dr < 0.$$

Note that (H_L) and (H_S) are complementary properties. We will show that a nonconstant radial stable solution of (1.1) satisfying (H_L) is *large* (i.e.

satisfies (L)) while a nonconstant radial stable solution of (1.1) satisfying (H_S) is *small* (i.e. satisfies (S)).

2. PROOF OF THE MAIN RESULTS

The following lemma follows easily from the ideas of the proof of [1, Lem. 2.2], which was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in \mathbb{R}^N for $N \leq 7$ (see [9, Th. 10.10] and [2, Rem. 2.2] for more details).

Lemma 2.1. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$ and u be a radial stable solution of (1.1). Let $1 \leq r_1 < r_2 < \infty$ and $\eta \in C^{0,1}([r_1, r_2])$ such that ηu_r vanishes at $r = r_1$ and $r = r_2$. Then*

$$\int_{r_1}^{r_2} r^{N-1} u_r^2 \left(\eta'^2 - \frac{N-1}{r^2} \eta^2 \right) dr \geq 0.$$

Proof. First of all, note that we can extend the second variation of energy Q_u to the set of functions $v \in C^{0,1}(\mathbb{R}^N \setminus \overline{B_1})$ with compact support in $\mathbb{R}^N \setminus \overline{B_1}$, obtaining $Q_u(v) \geq 0$ for such functions v . Hence, if $r_1 > 1$, we can take the radial function $v = \eta u_r \chi_{B_{r_2} \setminus \overline{B_{r_1}}}$. In fact, by an approximative method, we can also take this function v in the case $r_1 = 1$.

On the other hand, differentiating (1.1) with respect to r , we have

$$-\Delta u_r + \frac{N-1}{r^2} u_r = f'(u) u_r, \quad \text{for all } r \geq 1.$$

Following the ideas of the proof of [1, Lem. 2.2], we can multiply this equality by $\eta^2 u_r$ and integrate by parts in the annulus of radii r_1 and r_2 to obtain

$$\begin{aligned} 0 &= \int_{B_{r_2} \setminus \overline{B_{r_1}}} \left(\nabla u_r \nabla (\eta^2 u_r) + \frac{N-1}{r^2} u_r \eta^2 u_r - f'(u) u_r \eta^2 u_r \right) dx \\ &= \int_{B_{r_2} \setminus \overline{B_{r_1}}} \left(|\nabla (\eta u_r)|^2 - f'(u) (\eta u_r)^2 \right) dx - \int_{B_{r_2} \setminus \overline{B_{r_1}}} u_r^2 \left(|\nabla \eta|^2 - \frac{N-1}{r^2} \eta^2 \right) dx \\ &= Q(\eta u_r \chi_{B_{r_2} \setminus \overline{B_{r_1}}}) - \omega_N \int_{r_1}^{r_2} r^{N-1} u_r^2 \left(\eta'^2 - \frac{N-1}{r^2} \eta^2 \right) dr. \end{aligned}$$

Using the stability of u the lemma follows. \square

Lemma 2.2. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$ and u be a nonconstant radial stable solution of (1.1). Then u_r vanishes at most in one value in $[1, +\infty)$.*

Proof. Suppose by contradiction that there exist $1 \leq r_1 < r_2 < \infty$ such that $u_r(r_1) = u_r(r_2) = 0$. Taking $\eta \equiv 1$ in the previous lemma, we obtain

$$\int_{r_1}^{r_2} r^{N-1} u_r^2 \left(-\frac{N-1}{r^2} \right) dr \geq 0.$$

Hence we conclude that $u_r \equiv 0$ in $[r_1, r_2]$, which clearly forces u is constant in $\mathbb{R}^N \setminus B_1$, a contradiction. \square

2.1. Large solutions. In this subsection we will prove that a nonconstant radial stable solution of (1.1) satisfying (H_L) is *large* (i.e. satisfies (L)).

Lemma 2.3. *Let $N \geq 2$ and u be a nonconstant radial stable solution of (1.1) satisfying (H_L) . Then there exist $a \geq 1$ and $K > 0$ such that*

$$\int_r^{2r} \frac{ds}{u_r(s)^2} \leq K r^{N-2\sqrt{N-1}-1} \quad \forall r \geq a.$$

Proof. Consider $R_1 \geq 1$ of (H_L) . From Lemma 2.2 we can choose $a > R_1$ such that u_r does not vanish in $[a, \infty)$. We now fix $r \geq a$ and consider the function

$$\eta(t) = \begin{cases} a^{-\sqrt{N-1}} & \text{if } R_1 \leq t < a, \\ t^{-\sqrt{N-1}} & \text{if } a \leq t < r, \\ \frac{r^{-\sqrt{N-1}}}{\int_r^{2r} \frac{ds}{u_r(s)^2}} \int_t^{2r} \frac{ds}{u_r(s)^2} & \text{if } r \leq t \leq 2r. \end{cases}$$

Applying (H_L) (with $R_2 = 2r$) we have

$$\begin{aligned} 0 &\leq \int_{R_1}^{R_2} t^{N-1} u_r(t)^2 \left(\eta'(t)^2 - \frac{N-1}{t^2} \eta(t)^2 \right) dt \\ &= -(N-1) a^{-2\sqrt{N-1}} \int_{R_1}^a t^{N-3} u_r(t)^2 dt + \int_r^{2r} t^{N-1} u_r(t)^2 \left(\eta'(t)^2 - \frac{N-1}{t^2} \eta(t)^2 \right) dt \\ &\leq -(N-1) a^{-2\sqrt{N-1}} \int_{R_1}^a t^{N-3} u_r(t)^2 dt + (2r)^{N-1} \int_r^{2r} u_r(t)^2 \eta'(t)^2 dt \\ &= -(N-1) a^{-2\sqrt{N-1}} \int_{R_1}^a t^{N-3} u_r(t)^2 dt + (2r)^{N-1} \frac{r^{-2\sqrt{N-1}}}{\int_r^{2r} \frac{ds}{u_r(s)^2}}. \end{aligned}$$

This gives

$$(N-1) a^{-2\sqrt{N-1}} \int_{R_1}^a t^{N-3} u_r(t)^2 dt \leq 2^{N-1} \frac{r^{N-2\sqrt{N-1}-1}}{\int_r^{2r} \frac{ds}{u_r(s)^2}},$$

which is the desired conclusion for

$$K = 2^{N-1} / \left((N-1) a^{-2\sqrt{N-1}} \int_{R_1}^a t^{N-3} u_r(t)^2 dt \right).$$

\square

Lemma 2.4. *Let $N \geq 2$ and u be a nonconstant radial stable solution of (1.1) satisfying (H_L) . Then there exist $a \geq 1$ and $M' > 0$ such that*

$$|u(2r) - u(r)| \geq M' r^{-N/2 + \sqrt{N-1} + 2} \quad \forall r \geq a.$$

Proof. Take the same constant $a \geq 1$ of Lemma 2.3. Fix $r \geq a$ and consider the functions:

$$\alpha(s) = |u_r(s)|^{-\frac{2}{3}}, \quad s \in (r, 2r).$$

$$\beta(s) = |u_r(s)|^{\frac{2}{3}}, \quad s \in (r, 2r).$$

By Lemma 2.3 we have

$$\|\alpha\|_{L^3(r, 2r)} \leq K^{\frac{1}{3}} r^{\frac{N-2\sqrt{N-1}-1}{3}}$$

for a constant $K > 0$ not depending on $r \geq a$. On the other hand, since u_r does not vanish in $[a, \infty)$, it follows

$$\|\beta\|_{L^{3/2}(r, 2r)} = |u(2r) - u(r)|^{\frac{2}{3}}.$$

Applying Hölder inequality to functions α and β we deduce

$$r = \int_r^{2r} \alpha(s) \beta(s) ds \leq \|\alpha\|_{L^3(r, 2r)} \|\beta\|_{L^{3/2}(r, 2r)} \leq K^{\frac{1}{3}} r^{\frac{N-2\sqrt{N-1}-1}{3}} |u(2r) - u(r)|^{\frac{2}{3}},$$

which is the desired conclusion for $M' = K^{-1/2}$. \square

Proposition 2.5. *Let $N \geq 2$ and u be a nonconstant radial stable solution of (1.1) satisfying (H_L) . Then u satisfies (L) .*

Proof. Consider the numbers $a \geq 1$ and $M' > 0$ of Lemma 2.4. The proof will be divided into three cases:

- Case $2 \leq N \leq 9$.

It is easily seen that for every $r \geq a$ there exist an integer $m \geq 0$ and $a \leq z < 2a$ such that $r = 2^m z$. Thus, from Lemma 2.4 and the monotonicity of u in $[a, \infty)$, it follows that

$$\begin{aligned} |u(r)| &\geq |u(r) - u(z)| - |u(z)| = \sum_{k=1}^m |u(2^k z) - u(2^{k-1} z)| - |u(z)| \\ &\geq M' \sum_{k=1}^m (2^{k-1} z)^{-N/2 + \sqrt{N-1} + 2} - |u(z)| \\ &= M' \left(\frac{r^{-N/2 + \sqrt{N-1} + 2} - z^{-N/2 + \sqrt{N-1} + 2}}{2^{-N/2 + \sqrt{N-1} + 2} - 1} \right) - |u(z)|, \end{aligned}$$

where $M' > 0$ does not depend on $r \geq a$. Since $z \in [a, 2a)$, u is continuous and $-N/2 + \sqrt{N-1} + 2 > 0$, the above inequality is of the type

$$|u(r)| \geq M_1 r^{-N/2 + \sqrt{N-1} + 2} - M_2 \quad \forall r \geq a,$$

for certain $M_1, M_2 > 0$. It follows easily (L) in this case.

- Case $N = 10$.

In this case $-N/2 + \sqrt{N-1} + 2 = 0$. Following the same notation of the previous case, we can apply the same reasoning and conclude that

$$|u(r)| \geq M' m - |u(z)| = \frac{M'(\log r - \log z)}{\log 2} - |u(z)|,$$

and (L) follows immediately for this case.

- Case $N \geq 11$.

From Lemma 2.2 we deduce that there exists $u_\infty := \lim_{r \rightarrow \infty} u(r) \in [-\infty, +\infty]$. If $u_\infty = \pm\infty$ then the inequality in (L) is trivial. Then without loss of generality we can assume that $u_\infty \in \mathbb{R}$. Let $r \geq a$. From Lemma 2.4 and the monotonicity of u in $[a, \infty)$ we see that

$$\begin{aligned} |u_\infty - u(r)| &= \sum_{k=1}^{\infty} |u(2^k r) - u(2^{k-1} r)| \geq M' \sum_{k=1}^{\infty} (2^{k-1} r)^{-N/2 + \sqrt{N-1} + 2} \\ &= \left(M' \sum_{k=1}^{\infty} 2^{(k-1)(-N/2 + \sqrt{N-1} + 2)} \right) r^{-N/2 + \sqrt{N-1} + 2}. \end{aligned}$$

Finally, since $-N/2 + \sqrt{N-1} + 2 < 0$, the above series is convergent and (L) is proved in this case with $r_0 = a$. \square

2.2. Small solutions. In this subsection we will prove that a radial stable solution of (1.1) satisfying (H_S) is *small* (i.e. satisfies (S)).

Lemma 2.6. *Let $N \geq 2$ and u be a radial stable solution of (1.1) satisfying (H_S) . Then there exists $K > 0$ such that*

$$\int_r^{2r} u_r(s)^2 ds \leq K r^{-N-2\sqrt{N-1}+3} \quad \forall r \geq 1.$$

Proof. Take an arbitrary $r \geq 2$ and consider $R_1 = 2r$ in (H_S) . Then there exist $R_2 > 2r$ and $\eta_0 \in C^{0,1}([2r, R_2])$ such that $\eta_0(R_2) = 0$ and

$$(2.1) \quad \int_{2r}^{R_2} t^{N-1} u_r^2(t) \left(\eta_0'(t)^2 - \frac{N-1}{t^2} \eta_0(t)^2 \right) dt < 0.$$

Note that $\eta_0(2r) \neq 0$ (otherwise we would obtain a contradiction with Lemma 2.1 for $\eta = \eta_0$ and $[r_1, r_2] = [2r, R_2]$). Thus, multiplying by a constant if necessary, there is no loss of generality in assuming $\eta_0(2r) = r^{\sqrt{N-1}}$. We now fix $r \geq 2$ and consider the function

$$\eta(t) = \begin{cases} 2^{\sqrt{N-1}}(t-1) & \text{if } 1 \leq t < 2, \\ t^{\sqrt{N-1}} & \text{if } 2 \leq t < r, \\ r^{\sqrt{N-1}} & \text{if } r \leq t < 2r, \\ \eta_0(t) & \text{if } 2r \leq t \leq R_2. \end{cases}$$

Applying (2.1) and Lemma 2.1 to this function $\eta \in C^{0,1}([1, R_2])$ we have

$$\begin{aligned} 0 &\leq \int_1^{R_2} t^{N-1} u_r(t)^2 \left(\eta'(t)^2 - \frac{N-1}{t^2} \eta(t)^2 \right) dt \\ &= 4^{\sqrt{N-1}} \int_1^2 t^{N-1} u_r(t)^2 \left(1 - \frac{(N-1)(t-1)^2}{t^2} \right) dt - (N-1) r^{2\sqrt{N-1}} \int_r^{2r} t^{N-3} u_r(t)^2 dt \\ &\quad + \int_{2r}^{R_2} t^{N-1} u_r(t)^2 \left(\eta_0'(t)^2 - \frac{N-1}{t^2} \eta_0(t)^2 \right) dt \\ &< 4^{\sqrt{N-1}} \int_1^2 t^{N-1} u_r(t)^2 \left(1 - \frac{(N-1)(t-1)^2}{t^2} \right) dt - (N-1) r^{2\sqrt{N-1}} C_N r^{N-3} \int_r^{2r} u_r(t)^2 dt, \end{aligned}$$

where $C_N = \min\{1, 2^{N-3}\}$. This gives

$$(N-1) r^{2\sqrt{N-1}} C_N r^{N-3} \int_r^{2r} u_r(t)^2 dt < 4^{\sqrt{N-1}} \int_1^2 t^{N-1} u_r(t)^2 \left(1 - \frac{(N-1)(t-1)^2}{t^2} \right) dt,$$

which is our claim (if $r \geq 2$) for

$$K = 4^{\sqrt{N-1}} \int_1^2 t^{N-1} u_r(t)^2 \left(1 - \frac{(N-1)(t-1)^2}{t^2} \right) dt / ((N-1) C_N).$$

Finally, if $1 \leq r < 2$, since u_r is bounded in the interval $[1, 4]$, we also have the desired inequality and the lemma follows easily. \square

Lemma 2.7. *Let $N \geq 2$ and u be a radial stable solution of (1.1) satisfying (H_S) . Then there exists $M' > 0$ such that*

$$|u(2r) - u(r)| \leq M' r^{-N/2 - \sqrt{N-1} + 2} \quad \forall r \geq 1.$$

Proof. Fix $r \geq 1$. Applying Lemma 2.6 and Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
|u(2r) - u(r)| &= \left| \int_r^{2r} u_r(s) ds \right| \leq \int_r^{2r} |u_r(s)| ds \leq \left(\int_r^{2r} u_r(s)^2 ds \right)^{1/2} \left(\int_r^{2r} ds \right)^{1/2} \\
&\leq K^{\frac{1}{2}} r^{\frac{-N-2\sqrt{N-1}+3}{2}} r^{\frac{1}{2}},
\end{aligned}$$

which is our assertion for $M' = K^{1/2}$. \square

Proposition 2.8. *Let $N \geq 2$ and u be a radial stable solution of (1.1) satisfying (H_S) . Then u satisfies (S) .*

Proof. The proof will be divided into two cases:

- Case $N = 2$.

In this case $-N/2 - \sqrt{N-1} + 2 = 0$. Let $r \geq 1$. Then there exist an integer $m \geq 0$ and $1 \leq z < 2$ such that $r = 2^m z$. Thus, from Lemma 2.7 it follows that

$$\begin{aligned}
|u(r)| &\leq |u(r) - u(z)| + |u(z)| \leq \sum_{k=1}^m |u(2^k z) - u(2^{k-1} z)| + |u(z)| \leq M' m + |u(z)| \\
&= \frac{M'(\log r - \log z)}{\log 2} + |u(z)|,
\end{aligned}$$

where $M' > 0$ does not depends on $r \geq 1$. Since $z \in [1, 2)$ and u is continuous, then (S) follows immediately for this case.

- Case $N \geq 3$.

Let $r \geq 1$ and $j \in \mathbb{N}$ be arbitrary. Using Lemma 2.7 we obtain

$$\begin{aligned}
|u(2^j r) - u(r)| &\leq \sum_{k=1}^j |u(2^k r) - u(2^{k-1} r)| \leq M' \sum_{k=1}^j (2^{k-1} r)^{-N/2 - \sqrt{N-1} + 2} \\
&= \left(M' \sum_{k=1}^j 2^{(k-1)(-N/2 - \sqrt{N-1} + 2)} \right) r^{-N/2 - \sqrt{N-1} + 2}.
\end{aligned}$$

Since $-N/2 - \sqrt{N-1} + 2 < 0$ the above series is convergent and we can let $j \rightarrow \infty$ (remember that the existence of $u_\infty := \lim_{r \rightarrow \infty} u(r) \in [-\infty, +\infty]$ is guaranteed by Lemma 2.2) to obtain

$$|u_\infty - u(r)| \leq \frac{M'}{1 - 2^{-N/2 - \sqrt{N-1} + 2}} r^{-N/2 - \sqrt{N-1} + 2},$$

and the proof is complete in this case. \square

2.3. Proofs of Theorems 1.1 and 1.2 and Corollaries 1.3 and 1.4.

In this subsection we use the obtained results for *large* and *small* solutions to prove the main results of this paper.

Proof of Theorem 1.1. Let $N \geq 2$, $f \in C^1(\mathbb{R})$ and u be a radial stable solution of (1.1) (not necessarily bounded). By Lemma 2.2 it is immediate that there exists $u_\infty := \lim_{r \rightarrow \infty} u(r) \in [-\infty, +\infty]$.

If u is constant, then clearly u satisfies (S) and not (L). Hence, from the rest of the proof we will suppose that u is not constant. Obviously (H_L) and (H_S) are complementary properties, i.e. u satisfies either (H_L) or (H_S) . By Proposition 2.5 if u satisfies (H_L) then u satisfies (L) and by Proposition 2.8 if u satisfies (H_S) then u satisfies (S).

To finish the proof, let us observe that conditions (L) and (S) are clearly incompatible. \square

Proof of Theorem 1.2. Let $N \geq 3$, $f \in C^1(\mathbb{R})$ and u be a bounded radial stable solution of (1.1). By Theorem 1.1 we have that there exists $u_\infty := \lim_{r \rightarrow \infty} u(r) \in \mathbb{R}$. If u is constant, then clearly u satisfies (S') and not (L'). Hence, from the rest of the proof we will suppose that u is not constant.

Suppose that u satisfies (H_L) . We will show that u satisfies (L'). Since u satisfies (L) (again from Proposition 2.5) and $-N/2 + \sqrt{N-1} + 2 > 0$ for $3 \leq N \leq 9$, we deduce $N \geq 11$. What is left to show in this case is that $|\nabla u| \notin L^p(\mathbb{R}^N \setminus \overline{B_1})$ for every $1 \leq p \leq N/(N/2 - \sqrt{N-1} - 1)$. To this end, take the constant $a \geq 1$ of Lemma 2.3, fix $r \geq a$ and consider the functions

$$\alpha(s) = |u_r(s)|^{-\frac{2p}{p+2}}, \quad s \in (r, 2r).$$

$$\beta(s) = |u_r(s)|^{\frac{2p}{p+2}}, \quad s \in (r, 2r).$$

Applying Lemma 2.3 and Hölder inequality to functions α and β we deduce

$$\begin{aligned} r &= \int_r^{2r} \alpha(s)\beta(s)ds \leq \|\alpha\|_{L^{\frac{p+2}{p}}(r, 2r)} \|\beta\|_{L^{\frac{p+2}{2}}(r, 2r)} \\ &\leq K^{\frac{p}{p+2}} r^{\frac{p}{p+2}(N-2\sqrt{N-1}-1)} \left(\int_r^{2r} |u_r(s)|^p ds \right)^{\frac{2}{p+2}}, \end{aligned}$$

for a constant $K > 0$ not depending on $r \geq a$.

This gives

$$\int_r^{2r} |u_r(s)|^p ds \geq K^{-\frac{p}{2}} r^{p(-N/2 + \sqrt{N-1} + 1) + 1},$$

for $r \geq a$. From this, we obtain

$$\int_r^{2r} s^{N-1} |u_r(s)|^p ds \geq r^{N-1} \int_r^{2r} |u_r(s)|^p ds \geq K^{-\frac{p}{2}} r^{p(-N/2 + \sqrt{N-1} + 1) + N},$$

for $r \geq a$. Finally, since $N \geq 11$, $1 \leq p \leq N/(N/2 - \sqrt{N-1} - 1)$ and $r \geq a \geq 1$ we deduce that $r^{p(-N/2 + \sqrt{N-1} + 1) + N} \geq 1$, which implies $\int_r^{2r} s^{N-1} |u_r(s)|^p ds \geq K^{-p/2}$ for a constant $K > 0$ not depending on $r \geq a$. We conclude that $s^{N-1} |u_r(s)|^p \notin L^1(a, \infty)$ which clearly forces $|\nabla u| \notin L^p(\mathbb{R}^N \setminus \overline{B_1})$.

Suppose now that u satisfies (H_S) . We will show that u satisfies (S') . Since u satisfies (S) (again from Proposition 2.8), what is left to show in this case is that $|\nabla u| \in L^p(\mathbb{R}^N \setminus \overline{B_1})$ for every $N/(N/2 + \sqrt{N-1} - 1) < p \leq \infty$. For this purpose, let us observe that from standard regularity theory, since u is bounded we have that $|\nabla u|$ is also bounded. Then, by interpolation, it suffices to prove our claim for $N/(N/2 + \sqrt{N-1} - 1) < p < 2$. To this end consider $r \geq 1$. Applying Lemma 2.6 and Hölder inequality to functions $|u_r|^p$ and constant 1 and conjugate exponents $2/p$ and $2/(2-p)$ we deduce

$$\begin{aligned} \int_r^{2r} s^{N-1} |u_r(s)|^p ds &\leq (2r)^{N-1} \int_r^{2r} |u_r(s)|^p ds \\ &\leq (2r)^{N-1} \left(\int_r^{2r} |u_r(s)|^2 ds \right)^{p/2} \left(\int_r^{2r} ds \right)^{(2-p)/2} \\ &\leq (2r)^{N-1} K^{\frac{p}{2}} r^{\frac{p}{2}(-N-2\sqrt{N-1}+3)} r^{\frac{2-p}{2}} = 2^{N-1} K^{\frac{p}{2}} r^{p(-N/2 - \sqrt{N-1} + 1) + N}, \end{aligned}$$

for a constant $K > 0$ not depending on $r \geq 1$. Applying this inequality to $r = 2^j$, where $j \geq 0$ is an integer, we obtain

$$\begin{aligned} \int_1^\infty s^{N-1} |u_r(s)|^p ds &= \sum_{j=0}^\infty \int_{2^j}^{2^{j+1}} s^{N-1} |u_r(s)|^p ds \\ &\leq 2^{N-1} K^{\frac{p}{2}} \sum_{j=0}^\infty 2^{j(p(-N/2 - \sqrt{N-1} + 1) + N)}. \end{aligned}$$

Finally, since $N \geq 3$ and $N/(N/2 + \sqrt{N-1} - 1) < p < 2$, then $p(-N/2 - \sqrt{N-1} + 1) + N < 0$, which implies that the above series is convergent. It follows that $s^{N-1} |u_r(s)|^p \in L^1(1, \infty)$ which clearly shows that $|\nabla u| \in L^p(\mathbb{R}^N \setminus \overline{B_1})$ and the claim is proved.

Again, to finish the proof, let us observe that conditions (L') and (S') are clearly incompatible. \square

Proof of Corollary 1.3. Let $N \geq 2$, $g \in C^1(\mathbb{R})$ and u be a radial solution of (1.2) which is stable outside a compact set (not necessarily bounded). Then, there exists $R_0 > 0$ such that u is stable in $\mathbb{R}^N \setminus B_{R_0}$. It follows easily that the function $w(x) := u(R_0 x)$ is an stable solution of $-\Delta w = R_0^2 g(w)$ in $\mathbb{R}^N \setminus B_1$ and we can apply Theorem 1.1 to w . The proof is complete by

observing that there exists $u_\infty := \lim_{r \rightarrow \infty} u(r) = \lim_{r \rightarrow \infty} w(r)$ and that u satisfies (L) if and only if w satisfies (L), while u satisfies (S) if and only if w satisfies (S). \square

Proof of Corollary 1.4. Applying Theorem 1.2, this follows by the same method as in Corollary 1.3. \square

3. OPTIMALITY OF THE MAIN RESULTS AND FINAL REMARKS

We will see that the main results obtained in this paper are optimal. To this end, for every $N \geq 2$, let us define a family $\{u_\alpha, \alpha \in \mathbb{R}\} \subset C^\infty(\mathbb{R}^N \setminus B_1)$ of radial functions as

$$(3.1) \quad \begin{aligned} u_\alpha(r) &= r^\alpha \quad \forall r \geq 1, & \text{if } \alpha \neq 0. \\ u_0(r) &= \log r \quad \forall r \geq 1. \end{aligned}$$

It is easily seen that u_α is a solution of (1.1) with $f = f_\alpha \in C^1(\mathbb{R})$ defined by

$$\begin{aligned} f_\alpha(s) &= \begin{cases} -\alpha(\alpha + N - 2)s^{1-2/\alpha} & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases} & \text{if } \alpha < 0; \\ f_\alpha(s) &= \begin{cases} -\alpha(\alpha + N - 2)s^{1-2/\alpha} & \text{if } s \geq 1 \\ (\alpha + N - 2)((2 - \alpha)s - 2) & \text{if } s < 1 \end{cases} & \text{if } \alpha > 0; \\ f_0(s) &= -(N - 2)e^{-2s} & \text{if } s \in \mathbb{R}. \end{aligned}$$

The following example shows that the exponents $-N/2 \pm \sqrt{N-1} + 2$ which appear in Theorem 1.1 are optimal.

Example 3.1. For $N \geq 2$ consider the family $\{u_\alpha, \alpha \in \mathbb{R}\}$ defined in (3.1). Then,

$$u_\alpha \text{ is stable} \Leftrightarrow \left(\alpha \geq -N/2 + \sqrt{N-1} + 2 \text{ or } \alpha \leq -N/2 - \sqrt{N-1} + 2 \right).$$

Proof. Consider the above-mentioned functions f_α , $\alpha \in \mathbb{R}$. We check at once that

$$f'_\alpha(u_\alpha(r)) = \frac{-(\alpha - 2)(\alpha + N - 2)}{r^2} \quad \forall r \geq 1, \alpha \in \mathbb{R}.$$

Consider now Hardy Inequality: $\int_{\mathbb{R}^N} ((N-2)^2/(4r^2))v^2 \leq \int_{\mathbb{R}^N} |\nabla v|^2$, for every $v \in C^1(\mathbb{R}^N)$ with compact support, for $N \geq 3$. It is well known that the coefficient $(N-2)^2/4$ is optimal. Moreover it follows easily that this coefficient is also optimal if we consider $v \in C^1(\mathbb{R}^N \setminus \overline{B_1})$ with compact support in $\mathbb{R}^N \setminus \overline{B_1}$, for $N \geq 2$. Hence, the stability of u_α is equivalent to

$$-(\alpha - 2)(\alpha + N - 2) \leq \frac{(N - 2)^2}{4},$$

which is equivalent to

$$\alpha \geq -N/2 + \sqrt{N-1} + 2 \quad \text{or} \quad \alpha \leq -N/2 - \sqrt{N-1} + 2.$$

□

Remark 1. As we have mentioned, the above example shows the optimality of Theorem 1.1. Moreover, taking this type of solutions, it is also possible to demonstrate the optimality of Theorem 1.2. To this purpose, consider $\alpha = -N/2 + \sqrt{N-1} + 2$ and $N \geq 11$. Then it is a simple matter to see that u_α is a bounded radial stable solution satisfying $|\nabla u| \in L^p(\mathbb{R}^N \setminus \overline{B_1})$ for every $\frac{N}{N/2 - \sqrt{N-1} - 1} < p \leq \infty$. This proves the optimality of Theorem 1.2 for solutions u satisfying (L') . To see the optimality of Theorem 1.2 for solutions u satisfying (S') consider $\alpha = -N/2 - \sqrt{N-1} + 2$ and $N \geq 3$. Then it is easily seen that u_α is a bounded radial stable solution satisfying $|\nabla u| \notin L^p(\mathbb{R}^N \setminus \overline{B_1})$ for every $1 \leq p \leq \frac{N}{N/2 + \sqrt{N-1} - 1}$.

Remark 2. Corollaries 1.3 and 1.4 are also optimal. To see this, let us observe that it is possible to extend the family $\{u_\alpha, \alpha \in \mathbb{R}\} \subset C^\infty(\mathbb{R}^N \setminus B_1)$ of radial functions defined by (3.1) to another family $\{\overline{u}_\alpha, \alpha \in \mathbb{R}\} \subset C^\infty(\mathbb{R}^N)$ of radial functions satisfying $\overline{u}_\alpha(r) = u_\alpha(r)$, for every $r \geq 1$, $\alpha \in \mathbb{R}$, such that \overline{u}_α is a solution of (1.2) for some $g = g_\alpha \in C^1(\mathbb{R})$. To see this consider the following functions:

If $\alpha < 0$, take a C^∞ radial function \overline{u}_α satisfying $\overline{u}_\alpha(r) = 2 - r^2$, if $r \in [0, 1/2]$, $\overline{u}_\alpha(r) = u_\alpha(r)$, if $r \in [1, \infty)$ and $\overline{u}_\alpha'(r) < 0$, if $r > 0$. Then \overline{u}_α is a solution of (1.2) for $g = g_\alpha \in C^1(\mathbb{R})$ defined by $g_\alpha(s) = 2N$, if $s > 2$; $g_\alpha(s) = -\Delta \overline{u}(\overline{u}^{-1}(s))$, if $0 < s \leq 2$; $g_\alpha(s) = 0$, if $s \leq 0$.

If $\alpha \geq 0$, take a C^∞ radial function \overline{u}_α satisfying $\overline{u}_\alpha(r) = r^2 - 1$, if $r \in [0, 1/2]$, $\overline{u}_\alpha(r) = u_\alpha(r)$, if $r \in [1, \infty)$ and $\overline{u}_\alpha'(r) > 0$, if $r > 0$. Then \overline{u}_α is a solution of (1.2) for $g = g_\alpha \in C^1(\mathbb{R})$ defined by $g_\alpha(s) = -2N$, if $s < -1$; $g_\alpha(s) = -\Delta \overline{u}(\overline{u}^{-1}(s))$, if $s \geq -1$.

We claim that \overline{u}_α is stable outside a compact set $K \subset \mathbb{R}^N$ if and only if $\alpha \geq -N/2 + \sqrt{N-1} + 2$ or $\alpha \leq -N/2 - \sqrt{N-1} + 2$. Indeed, the sufficient condition follows from Example 3.1 (we can take $K = \overline{B_1}$) and the necessary condition is deduced from Corollary 1.3.

Let us emphasize that, by a result of the author ([10]), the solutions \overline{u}_α are unstable in \mathbb{R}^N for $\alpha \leq -N/2 - \sqrt{N-1} + 2$.

Remark 3. Lemma 2.2 says that u_r vanishes at most in one value in $[1, +\infty)$, for every nonconstant radial stable solution of (1.1), where $N \geq 2$ and $f \in C^1(\mathbb{R})$. In fact it is possible to prove that if u satisfies (S) , then u_r does not vanish in $[1, +\infty)$. Indeed, suppose by contradiction that $u_r(r_1) = 0$

for some $1 \leq r_1 < \infty$. Since u satisfies (H_S) (from Propositions 2.5 and 2.8), we can take $R_1 = r_1$ and apply Lemma 2.1 to $\eta_0 \in C^{0,1}([R_1, R_2])$, obtaining a contradiction.

Remark 4. Let $N \geq 2$ and $p > 1$. In [7, Th. 5] it is stated that if $0 \neq u \in C^2(\mathbb{R}^N)$ is a radial solution of $-\Delta u = |u|^{p-1}u$ in \mathbb{R}^N which is stable outside a compact set of \mathbb{R}^N , only two cases occur:

- (a) $N \geq 3$, $p = \frac{N+2}{N-2}$, $u(r) = \epsilon \left(\frac{\lambda \sqrt{N(N-2)}}{\lambda^2 + r^2} \right)^{(N-2)/2}$
with $\lambda > 0$, $\epsilon \in \{-1, 1\}$.
- (b) $N \geq 11$, $p \geq p_c := \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}$, u is stable, $u(r) = \epsilon \alpha^{2/(p-1)} v(\alpha r)$
with $\alpha > 0$, $\epsilon \in \{-1, 1\}$. The profile v satisfies: $v(0) = 1$, $v > 0$, $v' < 0$ in \mathbb{R}^+ .

According to our classification of radial solutions which are stable outside a compact set (Corollary 1.3) we see at once that the solutions of (a) satisfy (S), while solutions of (b) satisfy (L).

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